“Paretian Egalitarianism and Transitivity”

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Paretian egalitarianism, which combines a Pareto efficiency condition with a conditional equality condition, is intransitive. We argue that this does not count against Paretian egalitarianism on the ground that there are good reasons for thinking that overall value can be intransitive when based on more than one basic value.

Notes to readers:

1. This is a very crude preliminary draft. It’s not really ready for public consumption. We will send Shlomi a revised version on March 8th. We strongly encourage you to wait for that version.
2. Almost all the paper is on the general question of whether overall value can be intransitive and how rational choice is possible in the context of cycles. The worry about Paretian egalitarianism motivated us to explore this issue, but the issue took on a life of its own. Hence, the paper does not address anything specific equality. Sorry!
3. We have not yet had time to address carefully the issues raised in Larry’s new book. This is highly relevant and we will definitely address them in future revisions. (Larry: Wait for the next version, but then please do pass on prods for us to consider specific arguments you give.)

# The Intransitivity of (Anonymous) Paretian Egalitarianism

Let us start with some definitions needed for the definition of Paretian Egalitarianism.

**Pareto Superiority**: Alternative x is *weakly Pareto superior* to alternative y just in case each person has at least as much benefits in x as in y. If, in addition, there is at least one person that has more benefits in x than in y, then x is (strictly) *Pareto superior*.

**Pareto Incomparability:** Alternative x is *Pareto incomparable* with alternative y just in case it is not weakly Pareto superior to y and y is not weakly Pareto superior to it.

We can now define the following condition on justice ranking relation:

**Strong Pareto**: If x is weakly Pareto superior to y, then x is as least as just as y, and if x is (strictly) Pareto superior to y, then x is more just than y.

**Weak Egalitarianism:** For any two alternatives that are Pareto incomparable, (1) if their two benefit distributions are equally equal, then they are equally just, and (2) if the benefit distribution of one is more equal than that of the other, then the former is more just.

**Paretian Egalitarianism:** The conjunction of Strong Pareto and Weak Egalitarianism.

Paretian Egalitarianism gives priority to Pareto efficiency over equality.

All our key points below apply to Paretian Egalitarianism, but for brevity, however, we shall focus on a closely related principle based on the following strengthened notion of Pareto superiority.

**Anonymous Pareto Superiority**: Alternative x is *anonymously weakly Pareto superior* to alternative y just in case, *for some permutation of the benefits in x*, each person has at least as much benefits in x as in y. If, in addition, there is at least one person that has more benefits in x than in y, then x is *anonymously* (strictly) *Pareto superior*.

**Anonymous Pareto Incomparability:** Alternative x is *anonymously Pareto incomparable* with alternative y just in case it is not anonymously weakly Pareto superior to y and y is not anonymously weakly Pareto superior to it.

For example, <1,3,6> is Pareto incomparable with <2,1,6>. Nonetheless, it is anonymously Pareto superior to the latter, because <3,1,6> is a permutation of the former and it is Pareto superior to <2,1,6>.

Consider the following conditions:

**Anonymous Strong Pareto**: If x is anonymously weakly Pareto superior to y, then x is as least as just as y, and if x is anonymously (strictly) Pareto superior to y, then x is more just than y.

**Very Weak Egalitarianism:** For any two alternatives that are anonymously Pareto incomparable, (1) if their two benefit distributions are equally equal, then they are equally just, and (2) if the benefit distribution of one is more equal than that of the other, then the former is more just.

**Anonymous Paretian Egalitarianism:** The conjunction of Anonymous Strong Pareto and Very Weak Egalitarianism.

Anonymous Paretian Egalitarian gives a greater role to Paretian considerations (although of an anonymous sort) and less role to equality.

Anonymous Paretian Egalitarianism (and Paretian Egalitarianism) violates the following condition (which is weaker than Transitivity) on justice rankings:

**Acyclicity**: If, for alternatives x1,….xn, x1 is more just than x2, x2 is more just than x3, …. and xn-1 is more just than xn, then xn is not more just than x1.

To see the violation, consider the following, where equality, for illustration will be measured by total shortfall from the average (different examples would be needed for different measures of equality):

1-5-12 > 1-5-6 [anonymously Pareto superior]

1-5-6 > 2-4-9 [anonymously Pareto incomparable and more equal (shortfall of 3 vs. 4)]

2-4-9 > 1-5-12 [anonymously Pareto incomparable and more equal (shortfall of 4 vs. 6)]

Does this rule out Anonymous Paretian Egalitarianism (and Paretian Egalitarianism)? We shall argue that it does not, on the ground that, when overall value (e.g., justice value) is based on more than one basic value, it need not be Transitive or even Acyclic. We turn to that general issue now.

Two fundamental questions in normative (e.g., prudential and moral) reasoning are how to rank alternatives and how to make rational choices from a set of feasible alternatives. We will explore a framework where both rankings and choices are based on pro tanto reasons derived from pro tanto basic values. Overall value, we argue, need not be transitive (or even acyclic), even if the basic values are. Moreover, rational choice is still possible in the presence of intransitivity.

# Basic Values

A basic value, when applicable, constitutes a pro tanto reason for preferring one alternative to another. A pro tanto reason has some force, which can be overridden by countervailing pro tanto reasons, but which prevails in overall reason in the absence of countervailing reasons.[[1]](#endnote-1) For example, when considering two different public policies, the basic value of equality may provide a pro tanto moral reason for preferring one policy, whereas the basic value of total welfare may provide a pro tanto moral reason for preferring the other policy. Similarly, when reflecting upon two different life styles, the basic value of intellectual merit may provide a pro tanto prudential reason for preferring one life style, whereas the basic value of having a good family life may provide a pro tanto prudential reason for preferring the other life style.

We shall assume that there is a finite set, B, of basic values B1,..,Bn, where n>1. We leave open, however, the content of these values. We shall write x≥y (B1) as shorthand for the claim that basic value B1 provides a pro tanto reason for x being at least as good as alternative y (where strict preference and indifference are defined correspondingly).

We do not assume that basic values are complete. That is, we do not assume that, relative to any basic value, the following condition must hold:

**Completeness: F**or any two alternatives, either the first is at least as good as the second, or the second is at least as good as the first.

Total welfare may be an example of a basic value that is complete, if we assume that welfare is cardinally interpersonally comparable. For any two alternatives, there will be a pro tanto reason derived from total welfare that is weakly in favor of one of the alternatives. But a basic value does not have to be complete; it may only provide a pro tanto reason for a subset of all possible pairwise comparisons. The basic value of Pareto efficiency is an illustration of an incomplete basic value. It only provides a pro tanto reason for preferring one alternative to the other when everyone is at least as well off in the first alternative. It is silent, when some are better off in one alternative and others are better off in the other.

It is also common to assume that, relative to any basic values (and values generally), at least the weakest of the following conditions must hold:

**Transitivity:** For any three alternatives, if x is at least as good as y, and y is at least as good as z, then x is at least as good as z.

**Consistency:** For any number, n, of alternatives, if x1 is at least as good as x2, x2 is at least as good as x3, …. and xn-1 is at least as good as xn, then xn is not better than x1. Moreover, if for some i, 1 ≤ i ≤ n-1, xi is better than xi+1, then xn is not at least as good as x1.

**Acyclicity:** For any number, n, of alternatives, if x1 is better than x2, x2 is better than x3, …. and xn-1 is better than xn, then xn is not better than x1.

Transitivity entails Consistency, which entails Acyclicity, but the reverse entailments do not hold.

Consistency is like Transitivity, except that it allows incomparability. Suppose, for example, x is better than y, y is better than z, but x is *incomparable* with z (i.e., x is neither at least as good as z, nor is z at least as good as x). This violates Transitivity, but satisfies Consistency.

Acyclicity is weaker than Consistency, since it is silent about the case where one of the rankings is merely as equally good (as opposed to better). Suppose, for example, x is equally good with y, y is better than z, and z is better than x. This violates Consistency but not Acyclicity.

We shall assume that each basic value is transitive. We should note, however, that we are not convinced that basic value need be even acyclic. In line with Sen (2002), we believe that any a priori consistency condition is questionable. We endorse the following condition for basic values:

**Ranking Independence:** The ranking of two alternatives does not have any a priori implications for how other alternatives are ranked.

If this condition is accepted, it follows immediately that Transitivity is not a priori condition of rationality. This is because it requires that, for example, the rankings of x as better than y and y as better than z have the implication that x has to be considered better than z.

The basic value of the will of the majority provides an example of a basic value that does not satisfy Transitivity, since it may generate cyclic rankings (e.g., 2/3 favor x over y, 2/3 favor over z, and 2/3 favor z over x). Some people may argue that this shows that the will of the majority should not be considered a relevant basic value. We disagree. It seems quite clear from the many democratic movements across the world that the will of the majority is a relevant consideration for most people. The fact that it sometimes may provide cyclic rankings of alternatives may clearly pose a problem for making rational choices (a topic that we will address below), but it should not be used as an argument for ignoring this dimension value when making an overall evaluation of an alternative.

In the following, we will focus on basic values that are transitive, even though everything we say also applies to basic values that do not satisfy this requirement. Importantly, however, we allow for basic values being incomplete, which has important implications both for the overall value ranking and for how to make rational choices.

Note to reader (for the incomplete draft): Our approach is related to Temkin’s Essentially Comparative View, and in future revisions we will explore that connection more carefully.

# Overall Value

We here consider how the overall ranking of alternatives is based on the basic values, where x≥y means that x is overall at least as good as y (and strict preference and indifference are defined accordingly).

We start out by considering the following entirely uncontroversial condition on the relationship between basic values and overall value:

**Weak Value Dominance:** If, for each basic value,one alternative is at least as good as another, then it is also so overall. Moreover, if, in addition, relative to at least one basic value, one alternative is better than another, then it is also so overall.

Weak Value Dominance ensures that overall value agrees with the basic values in the pairwise comparison in the special case where they all agree. We will show that if the basic values are complete and transitive, then Weak Value Dominance is compatible with overall value ranking also being so:

We start by noting:

**Observation 1**: If the basic values are complete and transitive, then there exists an overall value ranking that satisfies Weak Value Dominance, Completeness, and Transitivity.

An example of such an overall value ranking is one that ranks alternatives on the basis of a specified lexical ordering of the basic values: (1) If x is better than y according to the first basic value, then it is better. (2) If x is not better than y according to the first n basic values, and x is better than y according to the n+1-th basic value, then x is better than y. (3) If no basic value ranks x better than y, then x and y are equally good. This is clearly complete and transitive. It also satisfies Weak Value Dominance, since (a) if all basic values rank x and y equally good, then so does overall value, and (b) if all basic values rank x at least as good as y, and at least one basic values ranks x better than y, then so does overall value.

In what follows, we will assume that basic values are incomplete. Incomplete basic values do not necessarily pose a problem when combined with Weak Value Dominance, but they do so when combined with the slightly stricter requirement of Strong Value Dominance:

**Strong Value Dominance:** If, for each basic value,one alternative is equally as good as another, then it is also so overall. Moreover, if, relative to at least one basic value, one alternative is better than another, and, relative to no basic value is it worse than the other, then it is better overall.

This like Weak Value Dominance, except that it holds that an alternative is overall better than another whenever at least one basic value judges it better and all other basic values judge it at least as good (as opposed to also judging it better). If each basic value is complete, then the strong and weak dominance conditions are equivalent. If, however, the basic values are not complete, the stronger condition can have content when the weak one does not. Suppose, for example, that relative to one value, x is better than y, and that all other basic values deem the two incomparable. In this case, the strong condition entails that x is overall better (since it is better in some respects and worse in none), but the weak condition is silent (since the other basic values do not judge x at least as good as y).

We find Strong Value Dominance highly plausible. Weak Value Dominance (overall at least as good, if at least as good for each value) is uncontroversial. It is, however, extremely weak, and Strong Value Dominance extends it in plausible ways. Suppose, for example, that all basic values but one are complete and judge one alternative better than another. Suppose further that the remaining basic value is complete except that it is silent for this pair of alternatives. Surely, the first alternative is better overall. One isolated case of silence cannot block a near-consensus of a large number basic values. Strong Value Dominance, of course, requires more than this, but the basic idea is the same. Although conflicts among basic value may generate incomparability in overall value, the mere silence of some basic value does not. Otherwise overall value could become extremely weak (with few assessments).

We now note the well-known result showing how Strong Value Dominance and incompleteness may lead to cycles (e.g., Schumm 1987, Rabinowicz 2000):

**Observation 2**: If Strong Value Dominance holds, then, if basic values are not complete, overall value can be acyclic, even if the basic values are transitive.

To see this, suppose, for example, that there are just three basic values, and according to the first x is better than y, according to the second y is better than z, and according to the third z is better than x. Assume further that each value is silent about all other comparisons between x, y, and z. Each basic value is thus trivially transitive. Strong Value Dominance then entails that x is overall better than y, y is overall better than z, and z is overall better than x (since each is better in one respect and worse in none). The ranking is thus cyclic.

To summarize, we have shown that, even if basic values are transitive, then overall value can be cyclic, when basic values are incomplete.

Once can avoid this implication by denying that basic values can be incomplete or by denying Strong Dominance. We believe that it is more plausible to accept that overall value can be cyclic in such context. Moreover, there is an independent reason to hold this view. We believe that the following condition holds for overall value (as it does for basic values):

**Ranking Independence** (repeated)**:** The ranking of two alternatives does not have any implications for how other alternatives are ranked.

Let us now examine whether rational choice is possible when overall value is cyclic.

# Rational Choice: How to Live with Cycles

We have argued that basic values can justify an overall value ranking that is cyclic, and thus we consider overall value cycles an unavoidable part of rational normative reasoning. We now examine how rational choice from a set of feasible alternatives is possible in light cycles in overall value.

The standard approach to rational choice is to assume the following criterion:

**Strongly Maximizing Rational Choice**: For any feasible set of alternatives, a given alternative is a rational choice if and only if it has the best overall value (i.e., it is overall at least as good as all other feasible alternatives).

Where, however, overall value is incomplete, there may be no best alternative (e.g., for {x,y}, where the two are incomparable). Rational choice, however, is possible in such situations, and thus the above criterion is too strong. A weaker, and more plausible, criterion is:

**Maximizing Rational Choice (MRC)**: For any feasible set of alternatives, a given alternative is a rational choice if and only if it has maximal overall value (i.e., no feasible alternative has greater overall value).

MRC is the same as SMITH except that it allows that some other feasible alternatives may be incomparable with a rational choice (as opposed to being equally good or worse). To illustrate, consider the feasible set of three alternatives {x,y,z}, where x>y, y>z, and x>z. In this case, SMITH and MRC each implies that the only rational choice is x. If, on the other hand, the overall value ranking is x>y, y=z, and silence concerning the ranking of x and z, then SMITH holds that there is no rational choice because no alternative is overall at least as good as the other two), but MRC implies that x and z are each rational choices (since no alternative is better than either).

A well-known result is:

**Observation 3**: If the overall value ranking is cyclic, then a rational choice is not possible according to MRC.

To see this, consider the feasible set of {x,y,z}, where x>y, y>, z, and z>x. In this case, the subset of maximal alternatives is empty, and MRC deems no alternative to be a rational choice.

Note: In the next version we will discuss three rules in Schwartz (1986): Gotcha, Getcha, and Soco. I’m guessing that two of these are the Schwartz rule and the Smith rule.

An alternative approach to rational choice was proposed by Schwartz (1970, 1972, 1986). To introduce this criterion, we need to introduce the notion of a Schwartz set. For a given set of feasible alternatives, (1) a *Schwartz subset* is any subset of the feasible alternatives such that (a) no feasible alternative not in this subset is overall better than any alternative in the subset, and (b) there is no non-empty proper subset of satisfying (a), and (2) the *Schwartz set* is the union of all Schwartz subsets. The Schwartz set is always non-empty. The criterion can now be formulated as follows:

**SCHWARTZ:** For any feasible set of alternatives, a given alternative is a rational choice if and only if it is in the Schwartz set.

In the simples case, where the feasible set consists of {x y,z}, and x>y, y>z, and x>z (i.e., no cycles), SCHWARTZ identifies the same set of rational choices as MRC. In this case, there is only one Schwartz subset, namely {x}, and therefore SCHWARTZ deems only x a rational choice. Unlike MRC, however, SCHWARTZ, deems some choices rational when there is a cycle, such as when x>y, y>z, and x>z. In this case, there is only one Schwartz subset, containing all the feasible alternatives, and thus SCHWARTZ deems each a rational choice.

Another possibility was suggested by Smith (1973). Define a *Smith set* as the smallest (in terms of cardinality) non-empty subset of feasible alternatives such that each alternative in the subset is better than each feasible alternative not in it. Smith sets are always non-empty, always exist (since the feasible set is a Smith set if no proper subset is), and are unique. To see that they are unique, suppose that there are two Smith sets S1 and S2. If one is a proper subset of the other, then the other is not a Smith set, since it is not the smallest set having the required property. Suppose, then that neither is a proper subset of the other. This implies that there exists an alternative x that belongs to S1 but not S2 and an alternative y that belongs to S2 but not S1. From the fact that x belongs to S1 but not S2, it follows that x is better than y. Similarly, from the fact that y belongs to S2 but not S1, it follows that y is better than x. Hence, we have a contradiction, which shows that for any feasible set there is a unique Smith set.

Consider then:

**SMITH:** For any feasible set of alternatives, a given alternative is a rational choice if and only if it is in the Smith set.

The crucial (but not the only) difference between Schwartz criterion and the Smith criterion is this: The Schwartz criterion appeal to the smallest set (under set inclusion) for which *no* *excluded* alternatives is *better* than any *included* alternative, whereas the Smith criterion appeals to the smallest set for which *each included* is alternative is better than any *excluded* alternative.

Like SCHWARTZ, SMITH allows for rational choice in the context of cycles. For example, if the feasible set is {x,y,z} and x>y, y>z, and z>x, like SCHWARTZ, SMITH judges all feasible alternatives to be rational choices. The two approaches, however, differ in the case where x>y, y=z, and x=z. In this case, SCHWARTZ judges x and z as the only two rational choices, whereas SMITH judges all three alternatives to be rational choices.

We believe, however, that both these approaches should be rejected. To defend this view, let us introduce the following basic conditions on rational choice:

**Rational Choice (RC)**: For any feasible set containing x, if x is a rational choice, then there does not exist another rational choice that it is (1) maximally good, and (2) better than x.

**Irrational Choice (IRC)**: For any feasible set containing x, if it is *not* a rational choice, then there exists a rational choice that is (1) maximally good, and (2) better than x.

We find both these conditions highly plausible. RC appears entirely uncontroversial. It simply states that if an alternative x is a rational choice, then there is no other rational choice that is maximal and also better than x. The rationale for this is that, if this other alternative is also a rational choice and maximal, then there is no justification for choosing x, given that alternative y is better than x. We also find IRC convincing. It states that, if an alternative x is not a rational choice, then there exists a rational choice that is both maximal and better than x.

It turns out, however, that neither SCHWARTZ nor SMITH satisfy both of these conditions.

**Observation 4**: If the overall value ranking is incomplete, SMITH violates RC and SCHWARTZ violates IRC.

Let us first prove the result for SMITH. Consider the feasible set {x,y,z}, where x>y, y>z, and silence elsewhere. In this case, SMITH holds that each of the three alternatives is a rational choice (since any excluded alternative would fail to be worse than some included alternative). But this violates RC, since alternative y is worse than x, which is both maximal and deemed a rational choice.

In contrast, in the above choice situation, SCHWARTZ judges only x to be a rational choice (since: (1) if it is not a rational choice, then neither is y and hence neither is z, and no Schwartz set is empty, and (2) if it is a rational choice, then y and z are not, since no subset of a Schwartz subset is a Schwartz subset). But this violates IRC, since there is no rational choice that is better than z (x is the rational choice and overall value ranking is silent in the comparison between x and z).

We shall argue for the following approach to rational choice. Define the TV set as the set of all alternatives that are not worse than some maximal alternative. Consider now the following choice criterion:

**Not Worse than a Maximal (NWM):** For any feasible set of alternatives, a feasible alternative is a rational choice if and only if it is in the TV set.

It turns out that NWM is the only choice criterion that satisfies RC and IRC.

**Observation 5**: NWM is the only choice criterion that always selects a non-empty set of rational choices and satisfies RC and IRC.

To show that NWM always selects a non-empty set of rational choices and satisfies RC and IRC, consider any feasible set. If there is a maximal alternative in this feasible set, then this alternative is in the TV set. If there is no maximal alternative in this feasible set, then all alternatives are in the TV set. Hence, the TV set is always non-empty and, consequently, NWM always deems a non-empty set of feasible alternatives as rational choices.

Let us now show that NWM satisfies RC. NWM deems all maximal alternatives as rational choices. Consider any alternative x for which there does not exist a maximal alternative that is better than x. This alternative is in the TV, and thus in NWM’s set of rational choices, in line with RC. Finally, let us show that NWM satisfies IRC. Consider any alternative x for which there does exist a maximal alternative that is better than x. This alternative is not in the TV, and thus not deemed by NWM to be a rational choice, in line with IRC.

To show that no choice criterion other than NWM satisfies the conditions, suppose that a given choice criterion always selects a non-empty set of rational choices and satisfies RC and IRC. For a given feasible set, suppose that the choice criterion deems an alternative that is not in the NWM set to be a rational choice. It follows from the definition of the NWM set that this alternative is worse than a maximal alternative, and thus this violates RC. Next, suppose that the choice criterion judges an alternative that is in the NWM set not to be a rational choice. It follows from the definition of the NWM set that this alternative is not worse than any maximal alternative, and thus this violates IRC. Taken together, this shows that there are no choice criterion other than NWM that always selects a non-empty set of rational choices and satisfies RC and IRC.

Note to reader: In the final version of the paper, we plan to provide a more detailed defense of NVM, where we also will discuss the fact that it violates the standard contraction consistency requirement.

# Some Applications

Note to reader: In the final version of the paper, we plan to apply our framework to discuss some of the classical paradoxes in the normative literature, including the Repugnant Conclusion and Sen’s Liberal Paradox.

# Concluding Remarks

To be completed

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1. Basic values are, at least roughly, what Temkin (2012) calls “ideals”. [↑](#endnote-ref-1)